

Counting on Rectangular Areas

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Abstract

In the first section of this paper we prove a theorem for the number of columns of a rectangular area that are identical to the given one. A special case, concerning $(0, 1)$ -matrices, is also stated.

In the next section we apply this theorem to derive several combinatorial identities by counting specified subsets of a finite set. This means that the obtained identities will involve binomial coefficients only. We start with a simple equation which is, in fact, an immediate consequence of Binomial theorem, but it is derived independently of it. The second result concerns sums of binomial coefficients. In a special case we obtain one of the best known binomial identity dealing with alternating sums. Klee's identity is also obtained as a special case as well as some formulae for partial sums of binomial coefficients, that is, for the numbers of Bernoulli's triangle.

1 A counting theorem

The set of natural numbers $\{1, 2, \dots, n\}$ will be denoted by $[n]$, and by $|X|$ will be denoted the number of elements of the set X .

For the proof of the main theorem we need the following simple result:

$$\sum_I (-1)^{|I|} = 0, \quad (1)$$

where I run over all subsets of $[n]$ (empty set included). This may be easily proved by induction or using Binomial theorem. But the proof by induction makes all further investigations independent even of Binomial theorem.

Let A be an $m \times n$ rectangular matrix filled with elements which belong to a set Ω .

By the i -column of A we shall mean each column of A that is equal to $[c_1, c_2, \dots, c_m]^T$, where c_1, c_2, \dots, c_m of Ω are given. We shall denote the number of i -columns of A by $\nu_A(c)$ or simply by $\nu(c)$.

For $I = \{i_1, i_2, \dots, i_k\} \subset [m]$, by $A(I)$ will be denoted the maximal number of columns j of A such that

$$a_{ij} \neq c_j, \quad (i \in I).$$

We also define

$$A(\emptyset) = n.$$

Theorem 1. *The number $\nu(c)$ of i -columns of A is equal*

$$\nu(c) = \sum_I (-1)^{|I|} A(I), \quad (2)$$

where summation is taken over all subsets I of $[m]$.

Proof. Theorem may be proved by the standard combinatorial method, by counting the contribution of each column of A in the sum on the right side of (2).

We give here a proof by induction. First, the formula will be proved in the case $\nu(c) = 0$ and $\nu(c) = n$. In the case $\nu(c) = n$ it is obvious that for $I \neq \emptyset$ we have $A(I) = 0$, which implies

$$\sum_I (-1)^{|I|} A(I) = n + \sum_{I \neq \emptyset} (-1)^{|I|} A(I) = n.$$

In the case $\nu(c) = 0$ we use induction on n . If $n = 1$ then the matrix A has only one column, which is not equal c . It yields that there exists $i_0 \in \{1, 2, \dots, m\}$ such that $a_{i_0,1} \neq c_{i_0}$. Denote by I_0 the set of all such numbers. Then $A(I) = 1$ if and only if $I \subset I_0$. From this and (1) we obtain

$$\sum_I (-1)^{|I|} A(I) = \sum_{I \subset I_0} (-1)^{|I|} = 0.$$

Suppose now that the formula is true for matrices with n columns and that A has $n + 1$ -columns, and $\nu_A(c) = 0$. Omitting the first column, the matrix B with n columns remains. If I_0 is the same as in the case $n = 1$, then

$$\begin{aligned} \sum_I (-1)^{|I|} A(I) &= \sum_{I \not\subset I_0} (-1)^{|I|} A(I) + \sum_{I \subset I_0} (-1)^{|I|} A(I) = \\ &= \sum_{I \not\subset I_0} (-1)^{|I|} B(I) + \sum_{I \subset I_0} (-1)^{|I|} (B(I) + 1) = \\ &= \sum_I (-1)^{|I|} B(I) + \sum_{I \subset I_0} (-1)^{|I|} = 0, \end{aligned}$$

since the first sum is equal zero by the induction hypothesis, and the second by (1).

For the rest of the proof we use induction on n again. For $n = 1$ the matrix A has only one column which is either equal c or not. In both cases theorem is true, from the preceding.

Suppose that theorem holds for n , and that the matrix A has $n + 1$ columns. We may suppose that $\nu(c) \geq 1$. Omitting one of the i -columns we obtain the matrix B with n columns. By the induction hypothesis theorem is true for B .

On the other hand it is clear that $A(I) = B(I)$ for each nonempty subset I . Furthermore A has one i-column more than B , which implies

$$\begin{aligned}\nu(c) &= \nu_A(c) = \nu_B(c) + 1 = 1 + \sum_I (-1)^{|I|} B(I) = \\ &= 1 + n + \sum_{I \neq \emptyset} (-1)^{|I|} B(I) = 1 + n + \sum_{I \neq \emptyset} (-1)^{|I|} A(I).\end{aligned}$$

Thus

$$\nu(c) = \sum_I (-1)^{|I|} A(I),$$

and theorem is proved.

If the number $A(I)$ does not depend on elements of the set I , but only on its number $|I|$ then the equation(2) may be written in the form

$$\nu(c) = \sum_{i=0}^m (-1)^i \binom{m}{i} A(i), \quad (3)$$

where $|I| = i$.

Our object of investigation will be $(0, 1)$ matrices. Let c be the i- column of a such matrix A . Take $I_0 \subseteq [m]$, $|I_0| = k$ such that

$$c_i = \begin{cases} 1 & i \in I_0 \\ 0 & i \notin I_0 \end{cases} \quad (4)$$

Then the number $A(I)$ is equal to the number of columns of A having 0's in the rows labelled by the set $I \cap I_0$, and 1's in the rows labelled by the set $I \setminus I_0$. Suppose that the number $A(I)$ depends only on $|I \cap I_0|$, $|I \setminus I_0|$. If we denote $|I \cap I_0| = i_1$, $|I \setminus I_0| = i_2$, $A(I) = A(i_1, i_2)$, then (2) may be written in the form

$$\nu(c) = \sum_{i_1=0}^k \sum_{i_2=0}^{m-k} (-1)^{i_1+i_2} \binom{k}{i_1} \binom{m-k}{i_2} A(i_1, i_2). \quad (5)$$

2 Counting subsets of a finite set

Suppose that a finite set $X = \{x_1, x_2, \dots, x_n\}$ is given. Label by $1, 2, \dots, 2^n$ all subsets of X arbitrary and define an $n \times 2^n$ matrix A in the following way

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \text{ lies in the set labelled by } j \\ 0 & \text{otherwise} \end{cases}. \quad (6)$$

Take $I_0 \subseteq [n]$, $|I_0| = k$, and form the submatrix B of A consisting of those rows of A which indices belong to I_0 . Let c be arbitrary i-column of B . Define

$$\begin{aligned}I'_0 &= \{i \in I_0 : c_i = 1\}, \\ I''_0 &= \{i \in I_0 : c_i = 0\}.\end{aligned} \quad (7)$$

The number $\nu(c)$ is equal to the number of subsets that contain the set $\{x_i, i \in I'_0\}$, and do not intersect the set $\{x_i : i \in I''_0\}$. There are obviously

$$\nu(c) = 2^{n-k},$$

such sets.

Furthermore, if $I \subseteq I_0$ then the number $B(I)$ is equal to the number of subsets that contain the set $\{x_i : i \in I \cap I''_0\}$, and do not meet the set $\{x_i : i \in I \cap I'_0\}$. It is clear that there are

$$B(I) = 2^{n-|I|}$$

such subsets, so that the formula (2) may be applied. It follows

$$2^{n-k} = \sum_{i=0}^k (-1)^i \binom{k}{i} 2^{n-i}.$$

Thus we have

Proposition 2.1. *For each nonnegative integer k holds*

$$1 = \sum_{i=0}^k (-1)^i \binom{k}{i} 2^{k-i}.$$

Note 2.1. *The preceding equation is a trivial consequence of Binomial theorem. But here it is obtained independently of this theorem.*

The preceding Proposition shows that counting i -columns over all subsets of X always produce the same result.

We shall now make some restrictions on the number of subsets of X . Take $0 \leq m_1 \leq m_2 \leq n$ fixed, and consider the submatrix C of A consisting of rows whose indices belong to I_0 , and columns corresponding to those subsets of X that have m , $(m_1 \leq m \leq m_2)$ elements.

Let c be an i -column of C . Define $I'_0 = \{i \in I_0 : c_i = 1\}$, $|I'_0| = l$.

The number $\nu(c)$ is equal to the number of sets that contain $\{x_i : i \in I'_0\}$, and do not intersect the sets $\{x_i : i \in I_0 \setminus I'_0\}$. We thus have

$$\nu = \sum_{i=m_1-|I'_0|}^{m_2-|I'_0|} \binom{n-|I_0|}{i}.$$

On the other hand, for $I \subseteq I_0$ the number $C(I)$ corresponds to the number of sets that contain $\{x_i : i \in I \setminus I'_0\}$, and do not intersect $\{x_i : i \in I \cap I'_0\}$. Its number is equal

$$\sum_{i_3=m_1-|I \setminus I'_0|}^{m_2-|I \setminus I'_0|} \binom{n-|I|}{i_3}.$$

It follows that the formula (5) may be applied. We thus have

Proposition 2.2. For $0 \leq m_1 \leq m_2 \leq n$, and $0 \leq l \leq k$ holds

$$\sum_{i=m_1-l}^{m_2-l} \binom{n-k}{i} = \sum_{i_1=0}^l \sum_{i_2=0}^{k-l} \sum_{i_3=m_1-i_2}^{m_2-i_2} (-1)^{i_1+i_2} \binom{l}{i_1} \binom{k-l}{i_2} \binom{n-i_1-i_2}{i_3} \quad (8)$$

In the special case when one takes $k = l$, $m_1 = m_2 = m$ we obtain

Corollary 2.1. For arbitrary nonnegative integers m, n, k holds

$$\binom{n-k}{m-k} = \sum_{i=0}^k (-1)^i \binom{k}{i} \binom{n-i}{m}. \quad (9)$$

Note 2.2. The preceding is one of the best known binomial identities. It appears in the book [1] in many different forms.

Taking $m_1 = m_2 = m$, in (8) one gets

Corollary 2.2. For arbitrary nonnegative integer m, n, k, l , ($l \leq k$) holds

$$\binom{n-k}{m-l} = \sum_{i_1=0}^l \sum_{i_2=0}^{k-l} (-1)^{i_1+i_2} \binom{l}{i_1} \binom{k-l}{i_2} \binom{n-i_1-i_2}{m-i_2}, \quad (10)$$

For $l = 0$ we obtain

$$\binom{n-k}{m} = \sum_{i=0}^k (-1)^i \binom{k}{i} \binom{n-i}{m-i}, \quad (11)$$

which is only another form of (9).

Taking $n = 2k$, $l = k$ in (10) we obtain

$$\binom{k}{m-k} = \sum_{i_1=0}^k (-1)^{i_1} \binom{k}{i_1} \binom{2k-i_1}{m}.$$

Substituting $k - i_1$ by i we obtain

Corollary 2.3. Klee's identity, ([2], p.13)

$$(-1)^k \binom{k}{m-k} = \sum_{i=0}^k (-1)^i \binom{k}{i} \binom{k+i}{m}.$$

From (8) we may obtain different formulae for partial sums of binomial coefficients, that is, for the numbers of Bernoulli's triangle. For instance, taking $l = 0$, $m_1 = 0$, $m_2 = m$ we obtain

Corollary 2.4. For any $0 \leq m \leq n$ and arbitrary nonnegative integer k holds

$$\sum_{i=0}^m \binom{n}{i} = \sum_{i_1=0}^k \sum_{i_2=0}^{m-i_1} (-1)^{i_1} \binom{k}{i_1} \binom{n+k-i_1}{i_2}. \quad (12)$$

Note 2.3. *The number k in the preceding equation may be considered as a free variable that takes nonnegative integer values. Specially, for $k = 1$ the equation represents the standard recursion formula for the numbers of Bernoulli's triangle.*

Taking $k = l = m_1$, $m_2 = m$ one obtains

$$\sum_{i=0}^m \binom{n}{i} = \sum_{i_1=0}^k \sum_{i_2=k}^{m+k} (-1)^{i_1} \binom{k}{i_1} \binom{n+k-i_1}{i_2} \quad (13)$$

Note 2.4. *The formulae (12) and (13) differs in the range of the index i_2 .*

References

- [1] J. Riordan, Combinatorial Identities. New York: Wiley, 1979.